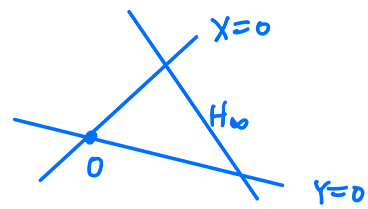
- $U_i := \{ [x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n \mid x_i \neq 0 \} \cong \mathbb{A}^n$

- $\mathbb{P}^n = \bigcup_{i=1}^{n+1} U_i$

Example: 1) $\mathbb{P}^1 \cong \mathbb{A}^1 \cup \{\infty\}$. $U_1 \cong \mathbb{A}^1$, $U_2 \cong (\mathbb{A}^1 \setminus \{0\}) \cup \{\infty\}$.

2) $\mathbb{P}^2 \cong \mathbb{A}^2 \cup H_\infty$

parallel lines ?



- V = irr. $\Leftrightarrow$ $I(V)$ = prime
- irr. decomposition.

projective variety := irreducible algebraic set in $\mathbb{P}^n$.

reduces questions from $\mathbb{P}^n$ ~~to~~ $\mathbb{A}^{n+1}$, for example:

projective nullstellensatz: $I \triangleleft k[x_1, \cdots, x_{n+1}]$ homg, Then

(1) $V_p(I) = \phi \Leftrightarrow I$ contains all forms of deg $\geq N$ for some N.

(2). $V_p(I) \neq \phi \Rightarrow I_p(V_p(I)) = \sqrt{I}$.

啥时非空

非空推广.

Pf. (1). $V_p(I) = \phi \Leftrightarrow V_a(I) \subseteq \{(0, \cdots, 0)\}$

$\Leftrightarrow (x_1, \cdots, x_{n+1}) \subseteq I_a(V_a(I)) = \sqrt{I}$

$\Leftrightarrow (x_1, \cdots, x_{n+1})^N \subseteq I$ for some N

(2) $I_p(V_p(I)) = I_a(C(V_p(I))) = I_a(V_a(I)) = \sqrt{I}$.

Cor : 1) $\{$radical homog. ideals $\neq m\} \xleftrightarrow{1:1} \{$alg. sets$\}$

2). $\{$homog. prime ideal $\neq m\} \xleftrightarrow{1:1} \{$proj. var$\}$

projective hypersurface $:= V(F)$ for some form F.

hyperplane

coordinate hyperplane

three coordinate axes in $\mathbb{P}^2$: $V(X_1), V(X_2), V(X_3)$.

$V =$ (nonempty) proj. vari in $\mathbb{P}^n$

$\Gamma_h(V) := k[X_1, \cdots, X_{n+1}]/I(V)$

homogeneous coordinate ring of V

$k_h(V) := \mathrm{Frac}(\Gamma_h(V))$ (V = irr.)

homogeneous function field of V

(V = affine alg. set. $\Rightarrow \forall f \in \Gamma(V)$ can be view as a function on V)

$P = [a_1 : a_2 : \cdots : a_{n+1}]$ $\forall F \in k[X_1, \cdots, X_{n+1}]$ $F(P) = ?$ not well-defined!

F & G forms of the same deg. $\Rightarrow \frac{F}{G}(P)$ can be defined at $G(P) \neq 0$.

$\Gamma = k[X_1 \cdots X_{n+1}]/I$

homog. ideal.

(nonzero) $f \in \Gamma$ is called a form of degree d, if $f = F \bmod I$ for some

6 form $F \in k[X_1, \cdots, X_{n+1}]$ of degree d.

Prop 1) $\forall f \in \Gamma$ written <u>uniquely</u> as $f = f_0 + \cdots + f_m$ with f_i a form of degree i

2) $\forall P \in V$, $\forall f, g \in \Gamma_h(V)$ forms of deg d. Then

$$f/g \text{ is a function on } V - \{P \mid g(P) = 0\}$$

Pf: ...

$$k(V) := \left\{ \frac{f}{g} \in k_h(V) \;\middle|\; f, g = \text{forms in } \Gamma_h(V) \text{ with the same degree}, g \neq 0 \right\}$$

Fact: subfield of $k_h(V)$ containing k.

rational functions on V.

$\forall P \in V$, $z \in k(V)$

z is defined at P if $z = \frac{f}{g}$ for some form f, g with $g(P) \neq 0$.

local ring of V at P

$$\mathcal{O}_P(V) := \{ z \in k(V) \mid z \text{ defined at } P \}$$

- $\mathcal{O}_P(V) \subset k(V)$ local subring

$$\mathfrak{m}_P(V) := \left\{ z = \frac{f}{g} \in k(V) \;\middle|\; g(P) \neq 0, f(P) = 0 \right\}$$

$$0 \to \mathfrak{m}_P(V) \rightarrowtail \mathcal{O}_P(V) \xrightarrow{z \mapsto z(P)} k \to 0$$

valuation map

Example: $V = \mathbb{P}^1$, $x := \frac{X}{Y}$, $y = \frac{Y}{X} = x^{-1} \in k(X,Y)$

$\Gamma_h(V) = k[X,Y]$ $\quad k_h(V) = k(X,Y)$

$\Gamma(V) \doteq ?$ $\quad k(V) = k(x)$

$V =$ affine alg. set $\Rightarrow \Gamma(V) = \{ f \in k(V) \mid f \text{ defined everywhere} \}$

$\Gamma(\mathbb{P}^1) = k$.